预期因子为反 Chaplygin 气体宏观生产模型黎曼解的渐近极限*

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摘　要：本文主要研究预期因子为反 Chaplygin 气体宏观生产模型黎曼解的极限行为．首先，研究模型的黎曼问题，得到 3 种黎曼解的结构：接触间断和疏散波组合($J_1 + R_2$)，接触间断和激波组合($J_1 + S_2$)，狄拉克激波(δS)．其次，研究反 Chaplygin 气体宏观生产模型的压力消失极限．当扰动参数 ε 减小到仅依赖初值的参数 ε_0 时，证明黎曼解($J_1 + S_2$)收敛到反 Chaplygin 气体状态方程的 δS．且当 ε 最终趋于 0 时，证明 δS 收敛到输运方程的 δS；此外，还证明黎曼解($J_1 + R_2$)收敛到输运方程的真空解．最后，给出具有代表性的实验结果数值．

关键词：宏观生产模型；反 Chaplygin 气体；黎曼问题；数值模拟

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The Asymptotic Limit of the Riemann Solution for the Macroscopic Production Model with Anti-Chaplygin Gas

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Abstract: This paper mainly studies the limit behavior of Riemann solutions for the macroscopic production model with anti-Chaplygin gas. Firstly, we investigate the Riemann problem associated with this model. Three types of Riemann solutions are obtained: a combination of contact discontinuity and rarefaction wave ($J_1 + R_2$), a combination of contact discontinuity and shock wave ($J_1 + S_2$), and Dirac shock wave (δS). Secondly, the pressure vanishing limit of the macroscopic production model of the anti-Chaplygin gas is studied. As the perturbation parameter ε decreases to the parameter ε_0, which is dependent only on the initial data, it is proved that the Riemann solution ($J_1 + S_2$) converges to the δS of the anti-Chaplygin gas state equation. Moreover, when ε eventually approaches 0, the δS converges to the δS of the transport equation. Additionally, it is proved that the Riemann solution ($J_1 + R_2$) converges to the vacuum solution of the transport equation. Finally, we present some representative numerical experimental results.

Key words: macroscopic production model; anti-Chaplygin gas; Riemann problem; numerical simulation

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0 引 言

类流体连续模型[1-2]是描述大批量产品流动的模型．一阶类流体模型主要由标量守恒律[3]组成，曾在初始阶段被用于生产流程的研究．为应对数据扩散，Forestier-Coste等[4]提出了二阶宏观生产模型：

$$\begin{cases}\rho_t + (\rho u)_x = 0,\\ \Big(\rho u\big(1 + p(\rho)\big)\Big)_t + \Big(\rho u^2\big(1 + p(\rho)\big)\Big)_x = 0,\end{cases} \tag{1}$$

式中：两个非负状态变量ρ和u分别表示产品的密度和速度；$x \in [0,\ 1]$和t表示生产的完成阶段和时间；状态方程$p(\rho) = \rho$表示生产线的预期因子，可以视为等温气体状态方程[5]．对于模型(1)，Sun[6]研究了黎曼问题，得到包含δ驻波的黎曼解．δ驻波用于解释生产链中因机器故障导致的瓶颈现象．随后，Zhang等[7]研究了黎曼解的渐近极限问题，发现当预期因子消失时，黎曼解不收敛于输运方程组的黎曼解．关于预期因子为等温气体宏观生产模型的更多研究，可参阅文献[8-9]．

当预期因子为等温气体时，预期因子p与密度ρ呈正比例关系，生产链中会出现瓶颈现象，进而导致生产效率降低、生产成本增加．本文旨在研究预期因子p与密度ρ呈反比例关系时，即：

$$p(\rho) = \frac{\varepsilon}{\rho}, \quad \varepsilon > 0, \tag{2}$$

生产链中会出现什么现象．状态方程(2)通常被称为反 Chaplygin 气体[10-12]，常用于解释宇宙中暗物质和暗能量的统一．

本文研究预期因子为反 Chaplygin 气体宏观生产模型：

$$\begin{cases}\rho_t + (\rho u)_x = 0,\\ (\rho u + \varepsilon u)_t + (\rho u^2 + \varepsilon u^2)_x = 0,\end{cases} \tag{3}$$

带有以下黎曼初值：

$$(\rho, u)(x, t = 0) = \begin{cases}(\rho_-,\ u_-), & -\infty < x < 0,\\ (\rho_+,\ u_+), & 0 < x < +\infty,\end{cases} \tag{4}$$

的问题，其中$\rho_\pm > 0$、$u_\pm > 0$．模型(3)具有2个不同的特征值$\lambda_1(\rho,\ u) = u$和$\lambda_2(\rho,\ u) = \left(1 + \dfrac{\varepsilon}{\rho + \varepsilon}\right)u$，故模型(3)是严格双曲的．$\lambda_1$对应的特征域是线性退化的，$\lambda_2$对应的特征域是真正非线性的．故$\lambda_1$对应的基本波为接触间断($J$)，$\lambda_2$对应的基本波为激波($S$)或稀疏波($R$)．利用相平面分析，黎曼问题(3)和(4)的解为：

(a)当$0 < u_- < u_+$时，接触间断和稀疏波组合($J_1 + R_2$)；

(b)当$0 < \dfrac{\rho_+ u_-}{\rho_+ + \varepsilon} < u_+ < u_-$时，接触间断和激波组合($J_1 + S_2$)；

(c)当$0 < u_+ \leqslant \dfrac{\rho_+ u_-}{\rho_+ + \varepsilon} < u_-$时，狄拉克激波($\delta S$)．

随后研究压力消失时黎曼解的渐近极限行为．当$\varepsilon \to 0$时，黎曼问题(3)和(4)的解收敛到输运方程：

$$\begin{cases}\rho_t + (\rho u)_x = 0,\\ (\rho u)_t + (\rho u^2)_x = 0,\end{cases} \tag{5}$$

的黎曼解．主要结论为以下定理．

定理 1 (i)对于情况$u_+ < u_-$，当ε减小到仅依赖初值的参数$\varepsilon_0 = \dfrac{\rho_+(u_- - u_+)}{u_+}$时，黎曼问题(3)和(4)的解($J_1 + S_2$)收敛到黎曼问题(3)和(4)的$\delta S$．当$\varepsilon$从$\varepsilon_0$减小到0时，黎曼问题(3)和(4)的$\delta S$收敛到黎曼问题(3)和(5)的$\delta S$．

(ii)对于情况$u_- < u_+$，当ε趋于0时，黎曼问题(3)和(4)的解($J_1 + R_2$)收敛到黎曼问题(4)和(5)的真空解．

本文主要利用压力消失极限方法研究带有反 Chaplygin 气体宏观生产模型黎曼解的渐近极限．2001年，Li[13]使用压力消失极限方法研究了等温可压缩欧拉方程组黎曼解的极限行为．2003年，Chen等[14]研究了等熵

欧拉方程组的压力消失极限问题，证明了该系统的黎曼解收敛到输运方程的狄拉克激波解和真空解，并将这一结果推广到非等熵欧拉方程组[15]．2010年，Shen等[16]研究了扰动Aw-Rascle交通流模型在压力消失极限下的质量集中和空穴现象．2014年，Yang等[17]研究了等熵修正Chaplygin气体欧拉方程组黎曼解的极限，证明了双参数压力消失时狄拉克激波和真空状态的形成．2022年，邵志强[18]研究了带有复合源项的可压缩欧拉方程组的黎曼问题，当绝热指数$\gamma \to 1$时，证明了黎曼解中集中现象和真空状态的形成．对于模型(1)，Chhatria等[19]研究了当预期因子为$p(\rho) = \kappa\left(\frac{\rho}{1 - b\rho}\right)^{\gamma}$时黎曼解的渐近极限问题．关于压力消失的其他相关研究可参阅文献[20-28]．

1 回 顾

本节主要回顾黎曼问题(4)和(5)的解．方程组(5)有1个二重特征值：

$$\lambda(\rho, u) = u, \tag{6}$$

对应的右特征向量为：

$$\vec{r} = (1, 0)^{T}, \tag{7}$$

由式(6)和(7)得：

$$\nabla\lambda \cdot \vec{r} = 0, \tag{8}$$

故特征域是线性退化的，其基本波为接触间断．根据u_+和u_-的大小关系，黎曼问题(4)和(5)的解分为以下3种情况．

当$u_+ > u_-$时，黎曼问题(4)和(5)的解为：

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_-), & x < u_- t, \\ \left(0, \dfrac{x}{t}\right), & u_- t \leqslant x \leqslant u_+ t, \\ (\rho_+, u_+), & x > u_+ t. \end{cases} \tag{9}$$

当$u_- = u_+$时，黎曼问题(4)和(5)的解为：

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_-), & x < u_- t, \\ (\rho_+, u_+), & x > u_+ t. \end{cases} \tag{10}$$

当$u_- > u_+$时，黎曼问题(4)和(5)的解为：

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_-), & x < u_\delta t, \\ (w(t)\delta(x - u_\delta t), u_\delta), & x = u_\delta t, \\ (\rho_+, u_+), & x > u_\delta t, \end{cases} \tag{11}$$

式中：$w(t)$和u_δ分别表示δ激波的强度和传播速度．u_δ和$w(t)$的表达式为：
当$\rho_+ \neq \rho_-$时，

$$\begin{cases} u_\delta = \dfrac{\sqrt{\rho_-}\, u_- + \sqrt{\rho_+}\, u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \\ w(t) = \sqrt{\rho_- \rho_+}\,(u_- - u_+)t. \end{cases} \tag{12}$$

当$\rho_+ = \rho_-$时，

$$\begin{cases} u_\delta = \dfrac{u_+ + u_-}{2}, \\ w(t) = \rho_+(u_+ - u_+)t. \end{cases} \tag{13}$$

详解请参阅文献[29]．

2 黎曼问题(3)和(4)的解

本节主要研究黎曼问题(3)和(4)的解．模型(3)有两个不同的实特征值：

$$\lambda_1(\rho,u)=u,\qquad \lambda_2(\rho,u)=\left(1+\frac{\varepsilon}{\rho+\varepsilon}\right)u, \tag{14}$$

对应的右特征向量为：

$$\vec{r}_1=(1,0)^T,\qquad \vec{r}_2=\left(\frac{\rho+\varepsilon}{\varepsilon u},\frac{1}{\rho}\right)^T, \tag{15}$$

由式(14)和(15)得：

$$\nabla\lambda_1\cdot\vec{r}_1=0,\qquad \nabla\lambda_2\cdot\vec{r}_2=\frac{2\varepsilon}{\rho(\rho+\varepsilon)}\neq 0, \tag{16}$$

故第一特征域是线性退化的,其基本波为接触间断．第二特征域是真正非线性的,其基本波分别为稀疏波或激波．

对于光滑解,寻找下列形式的自相似解：

$$(\rho,u)(x,t)=(\rho,u)(\xi),\quad \xi=\frac{x}{t}, \tag{17}$$

黎曼问题(3)和(4)可转化为无穷远边值问题：

$$\begin{cases}-\xi\rho_\xi+(\rho u)_\xi=0,\\ -\xi(\rho u+\varepsilon u)_\xi+\xi(\rho u^2+\varepsilon u^2)_\xi=0,\end{cases} \tag{18}$$

式中:$(\rho,\ u)(\pm\infty)=(\rho_\pm,\ u_\pm)$.

由式(18)得：

$$\begin{pmatrix}u-\xi & \rho\\ u^2-u\xi & 2\rho u+2u-\rho\xi-\varepsilon\xi\end{pmatrix}\begin{pmatrix}\rho\\ u\end{pmatrix}_\xi=\begin{pmatrix}0\\0\end{pmatrix}, \tag{19}$$

由式(19)得常状态解：

$$(\rho,u)(x,t)=const,\quad (\rho>0), \tag{20}$$

或稀疏波解：

$$R_2(\rho_-,u_-):\begin{cases}\xi=\lambda_2(\rho,u)=\left(1+\dfrac{\varepsilon}{\rho+\varepsilon}\right)u,\\ u\left(1+\dfrac{\varepsilon}{\rho}\right)=u_-\left(1+\dfrac{\varepsilon}{\rho_-}\right),\\ u_-<u,\ \rho>\rho_-.\end{cases} \tag{21}$$

由Rankine-Hugoniot条件得间断解．满足：

$$\begin{cases}\sigma[\rho]=[\rho u],\\ \sigma[\rho u+\varepsilon u]=[\rho u^2+\varepsilon u^2],\end{cases} \tag{22}$$

式中:$\sigma=\dfrac{\mathrm{d}x}{\mathrm{d}t}$; $[\rho]=\rho-\rho_-$．由式(22)得：

接触间断解：

$$J_1(\rho_-,u_-):\sigma_1=u=u_-. \tag{23}$$

激波解：

$$S_2(\rho_-,u_-):\begin{cases}\sigma_2=\dfrac{\rho u-\rho_-u_-}{\rho-\rho_-},\\ u\left(1+\dfrac{\varepsilon}{\rho}\right)=u_-\left(1+\dfrac{\varepsilon}{\rho_-}\right),\\ u<u_-,\ \rho<\rho_-.\end{cases} \tag{24}$$

给定一个左状态(ρ_-, u_-)，在相平面绘制接触间断J_1曲线$u = u_-$. 同时，绘制激波S_2和稀疏波R_2曲线$u\left(1 + \frac{\varepsilon}{\rho}\right) = u_-\left(1 + \frac{\varepsilon}{\rho_-}\right)$，该曲线单调递增，且以直线$u = \left(1 + \frac{\varepsilon}{\rho_-}\right)u_-$为渐近线. 此外，从点$\left(\rho_-, \frac{\rho_- u_-}{\rho_- + \varepsilon}\right)$出发，绘制曲线：

$$S_\gamma: u = \frac{\rho u_-}{\rho + \varepsilon}, \tag{25}$$

故相平面(ρ, u)被分成3个区域（图 1）：$\Omega_1 = \left\{(\rho, u) \middle| 0 < u_- < u_+\right\}$，$\Omega_2 = \left\{(\rho, u) \middle| 0 < \frac{\rho_+ u_-}{\rho_+ + \varepsilon} < u_+ < u_-\right\}$，$\Omega_3 = \left\{(\rho, u) \middle| 0 < u_+ \leqslant \frac{\rho_+ u_-}{\rho_+ + \varepsilon} < u_-\right\}$.

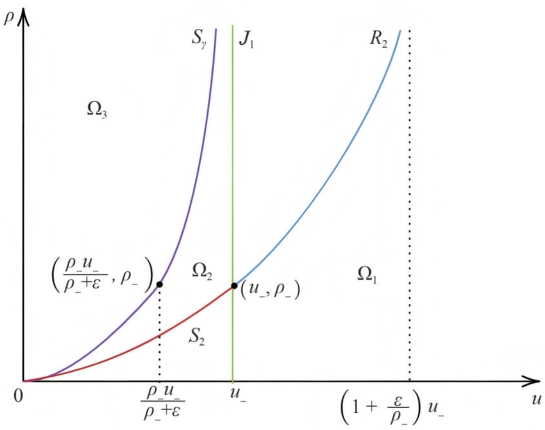

图 1　相平面(ρ, u)

Figure 1　Phase plane(ρ, u)

（a）当$(\rho_+, u_+) \in \Omega_1(\rho_-, u_-)$时，黎曼问题(3)和(4)的解为$(J_1 + R_2)$，即：

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_-), & x < \sigma_1 t, \\ (\rho_*, u_*), & \sigma_1 t < x < \lambda_2(\rho_*, u_*)t, \\ (\rho, u), & \lambda_2(\rho_*, u_*)t < x < \lambda_2(\rho_+, u_+)t, \\ (\rho_+, u_+), & x > \lambda_2(\rho_+, u_+)t, \end{cases} \tag{26}$$

其中(ρ_*, u_*)满足：

$$(\rho_*, u_*) = \left(\frac{\varepsilon \rho_+ u_-}{\varepsilon u_+ + \rho_+(u_+ - u_-)}, u_-\right), \tag{27}$$

(ρ, u)满足：

$$\begin{cases} \xi = \lambda_2(\rho, u) = \left(1 + \frac{\varepsilon}{\rho + \varepsilon}\right)u, \\ \left(1 + \frac{\varepsilon}{\rho}\right)u = \left(1 + \frac{\varepsilon}{\rho_+}\right)u_+. \end{cases} \tag{28}$$

（b）当$(\rho_+, u_+) \in \Omega_2(\rho_-, u_-)$时，黎曼问题(3)和(4)的解为$(J_1 + S_2)$，即：

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_-), & x < \sigma_1 t, \\ (\rho_*, u_*), & \sigma_1 t < x < \sigma_2 t, \\ (\rho_+, u_+), & x > \sigma_2 t, \end{cases} \tag{29}$$

其中(ρ_*, u_*)满足:

$$(\rho_*, u_*) = \left(\frac{\varepsilon\rho_+ u_-}{\varepsilon u_+ + \rho_+(u_+ - u_-)}, u_-\right), \tag{30}$$

接触间断J_1和激波S_2速度满足:

$$\sigma_1 = u_-, \qquad \sigma_2 = \frac{\rho_+ u_+ - \rho_* u_*}{\rho_+ - \rho_*}. \tag{31}$$

(c)当$(\rho_+, u_+) \in \Omega_3(\rho_-, u_-)$时,黎曼问题(3)和(4)的解为$\delta S$,即:

$$(\rho, u)(x,t) = \begin{cases} (\rho_-, u_-), & x < \tilde{u}_\delta t, \\ (\tilde{w}(t)\delta(x - \tilde{u}_\delta t), \tilde{u}_\delta), & x = \tilde{u}_\delta t, \\ (\rho_+, u_+), & x > \tilde{u}_\delta t, \end{cases} \tag{32}$$

式中:$\tilde{w}(t)$和$\tilde{u}_\delta$分别表示δ激波的强度和传播速度. $\tilde{w}(t)$和$\tilde{u}_\delta$表达式为:
当$\rho_+ \neq \rho_-$时,

$$\begin{cases} \tilde{u}_\delta = \dfrac{2(\rho_+ u_+ - \rho_- u_-) + \varepsilon(u_+ - u_-) + A}{2(\rho_+ - \rho_-)}, \\ \tilde{w}(t) = \left(\dfrac{2(\rho_+ u_+ - \rho_- u_-) + \varepsilon(u_+ - u_-) + A}{2} - (\rho_+ u_+ - \rho_- u_-)\right)t, \\ A = \sqrt{4\rho_+\rho_-(u_+ - u_-)^2 + 4\varepsilon(\rho_+ u_- - \rho_- u_+)(u_- - u_+) + \varepsilon^2(u_- - u_+)^2}. \end{cases} \tag{33}$$

当$\rho_+ = \rho_-$时,

$$\begin{cases} \tilde{u}_\delta = \dfrac{(\rho_+ + \varepsilon)(u_+ + u_-)}{2\rho_+ + \varepsilon}, \\ \tilde{w}(t) = \rho_+(u_- - u_+)t. \end{cases} \tag{34}$$

且$\tilde{u}_\delta$满足广义熵条件:

$$\lambda_1(\rho_+, u_+) < \lambda_2(\rho_+, u_+) \leqslant \tilde{u}_\delta \leqslant \lambda_1(\rho_-, u_-) < \lambda_2(\rho_-, u_-). \tag{35}$$

3 黎曼问题(3)和(4)解的渐近极限

本节利用压力消失极限方法研究黎曼问题(3)和(4)解的极限行为.

引理 1 若$u_- > u_+$、$\varepsilon > \varepsilon_0 > 0$,则$\lim\limits_{\varepsilon \to \varepsilon_0^+} \rho_* = +\infty$.

证明 若$u_- > u_+$、$(\rho_+, u_+) \in \Omega_2(\rho_+, u_-)$,则:

$$u_+ > \frac{\rho_+ u_-}{\rho_+ + \varepsilon}, \tag{36}$$

由式(36)得:

$$\varepsilon > \frac{\rho_+(u_- - u_+)}{u_+} =: \varepsilon_0, \tag{37}$$

由式(37)得:

$$\lim_{\varepsilon \to \varepsilon_0^+} \rho_* = \lim_{\varepsilon \to \varepsilon_0^+} \frac{\varepsilon\rho_+ u_-}{\varepsilon u_+ + \rho_+(u_+ - u_-)} = +\infty. \tag{38}$$

引理 2 当$u_- > u_+$、$\varepsilon > \varepsilon_0 = \dfrac{\rho_+(u_- - u_+)}{u_+} > 0$时,

$$\lim_{\varepsilon \to \varepsilon_0^+} u_* = \lim_{\varepsilon \to \varepsilon_0^+} \sigma_1 = \lim_{\varepsilon \to \varepsilon_0^+} \sigma_2 = u_- =: \widetilde{u}_\delta, \tag{39}$$

和

$$\lim_{\varepsilon \to \varepsilon_0^+} \int_{\sigma_1 t}^{\sigma_2 t} \rho_* \mathrm{d}x = \rho_+(u_- - u_+)t, \tag{40}$$

成立.

证明 由式(31)和(38)得:

$$\begin{cases} \lim\limits_{\varepsilon \to \varepsilon_0^+} u_* = u_-, \\ \lim\limits_{\varepsilon \to \varepsilon_0^+} \sigma_1 = u_-, \\ \lim\limits_{\varepsilon \to \varepsilon_0^+} \sigma_2 = \lim\limits_{\varepsilon \to \varepsilon_0^+} \dfrac{\rho_+ u_+ - \rho_* u_*}{\rho_+ - \rho_*} = u_* = u_-, \end{cases} \tag{41}$$

由式(41)可知,当$\varepsilon \to \varepsilon_0^+$时,激波$S_2$与接触间断$J_1$重合(图2).

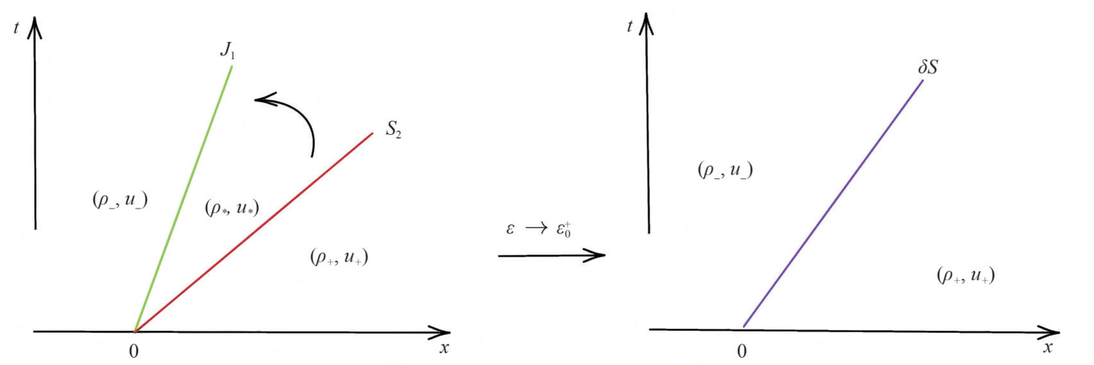

图 2 物理平面(x, t)

Figure 2 The physical plane(x, t)

由Rankine-Hugoniot条件得:

$$\begin{cases} \sigma_1(\rho_* - \rho_-) = \rho_* u_* - \rho_- u_-, \\ \sigma_2(\rho_+ - \rho_*) = \rho_+ u_+ - \rho_* u_*, \end{cases} \tag{42}$$

由式(42)可知:

$$\lim_{\varepsilon \to \varepsilon_0^+} (\sigma_2 - \sigma_1)\rho_* t = \left(\tilde{u}_\delta(\rho_+ - \rho_-) - (\rho_+ u_+ - \rho_- u_-)\right)t = \rho_+(u_- - u_+)t, \tag{43}$$

由式(43)得:

$$\lim_{\varepsilon \to \varepsilon_0^+} \int_{\sigma_1 t}^{\sigma_2 t} \rho_* \mathrm{d}x = \lim_{\varepsilon \to \varepsilon_0^+} (\sigma_2 - \sigma_1)\rho_* t = \rho_+(u_- - u_+)t. \tag{44}$$

定理 2 当$0 < u_+ < u_-$、$0 < \varepsilon_0 < \varepsilon$时,模型(3)和(4)的黎曼解在分布意义下收敛到(3)和(4)的狄拉克激波解. 且有:

$$\lim_{\varepsilon \to \varepsilon_0^+} \rho = \rho_- + (\rho_+ - \rho_-)H(x - u_- t) + \rho_+(u_- - u_+)t\delta(x - u_- t), \tag{45}$$

$$\lim_{\varepsilon \to \varepsilon_0^+} (\rho u + \varepsilon u) = \rho_- u_- + \varepsilon u_- + (\rho_+ u_+ + \varepsilon u_+ - \rho_- u_- - \varepsilon u_-)H(x - u_- t) + \rho_+ u_-(u_- - u_+)t\delta(x - u_- t), \tag{46}$$

式中:H和δ代表赫维赛德函数和狄拉克函数.

证明 当$0 < u_+ < u_-$、$0 < \varepsilon_0 < \varepsilon$时,令$\xi = \dfrac{x}{t}$,黎曼问题(3)和(4)的解为(29),对任意$\phi(\xi) \in C_0^\infty(-\infty, +\infty)$,其满足以下弱解定义:

$$\int_{-\infty}^{+\infty} (\xi - u(\xi))\rho(\xi)\phi'(\xi)\mathrm{d}\xi + \int_{-\infty}^{+\infty} \rho(\xi)\phi(\xi)\mathrm{d}\xi = 0, \tag{47}$$

$$\int_{-\infty}^{+\infty} (\xi - u(\xi))(\rho(\xi)u(\xi) + \varepsilon u(\xi))\phi'(\xi)\mathrm{d}\xi + \int_{-\infty}^{+\infty} (\rho(\xi)u(\xi) + \varepsilon u(\xi))\phi(\xi)\mathrm{d}\xi = 0. \tag{48}$$

式(47)左侧第1个积分可分解为:

$$\int_{-\infty}^{+\infty}\big(\xi-u(\xi)\big)\rho(\xi)\phi'(\xi)\mathrm{d}\xi=\left(\int_{-\infty}^{\sigma_1}+\int_{\sigma_1}^{\sigma_2}+\int_{\sigma_2}^{+\infty}\right)\big(\xi-u(\xi)\big)\rho(\xi)\phi'(\xi)\mathrm{d}\xi. \tag{49}$$

当$\varepsilon\to\varepsilon_0^+$时,$\lim\limits_{\varepsilon\to\varepsilon_0^+}\sigma_1=\lim\limits_{\varepsilon\to\varepsilon_0^+}\sigma_2=\lim\limits_{\varepsilon\to\varepsilon_0^+}u_*=u_\delta=u_-$,对式(49)的第1项和第3项取极限得:

$$\begin{aligned}&\lim_{\varepsilon\to\varepsilon_0^+}\left(\int_{-\infty}^{\sigma_1}+\int_{\sigma_2}^{+\infty}\right)(\xi-u(\xi))\rho(\xi)\phi'(\xi)\,\mathrm{d}\xi\\&=\lim_{\varepsilon\to\varepsilon_0^+}\int_{-\infty}^{\sigma_1}(\xi-u_-)\rho_-\phi'(\xi)\,\mathrm{d}\xi+\lim_{\varepsilon\to\varepsilon_0^+}\int_{\sigma_2}^{+\infty}(\xi-u_+)\rho_+\phi'(\xi)\,\mathrm{d}\xi\\&=\lim_{\varepsilon\to\varepsilon_0^+}\left(-\rho_-u_-\phi(\sigma_1)+\rho_+u_+(\sigma_2)\phi+\rho_-\sigma_1\phi(\sigma_1)-\rho_+\sigma_2\phi(\sigma_2)-\rho_-\int_{-\infty}^{\sigma_1}\phi(\xi)\,\mathrm{d}\xi-\rho_+\int_{\sigma_2}^{+\infty}\phi(\xi)\,\mathrm{d}\xi\right)\\&=\big([\rho u]-\tilde u_\delta[\rho]\big)\phi(\tilde u_\delta)-\int_{-\infty}^{+\infty}\big(\rho_-+[\rho]H(\xi-\tilde u_\delta)\big)\phi(\xi)\mathrm{d}\xi.\end{aligned} \tag{50}$$

当$\varepsilon\to\varepsilon_0^+$时,对式(49)的第2项取极限得:

$$\begin{aligned}&\lim_{\varepsilon\to\varepsilon_0^+}\int_{\sigma_1}^{\sigma_2}\big(\xi-u(\xi)\big)\rho(\xi)\phi'(\xi)\mathrm{d}\xi\\&=\lim_{\varepsilon\to\varepsilon_0^+}\int_{\sigma_1}^{\sigma_2}(\xi-u_*)\rho_*\phi'(\xi)\mathrm{d}\xi\\&=\lim_{\varepsilon\to\varepsilon_0^+}\rho_*(\sigma_2-\sigma_1)\left(\frac{\sigma_2\phi(\sigma_2)-\phi\sigma_1(\sigma_1)}{\sigma_2-\sigma_1}-u_*\frac{\phi(\sigma_2)-\phi(\sigma_1)}{\sigma_2-\sigma_1}-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\phi(\xi)\mathrm{d}\xi\right)\\&=\big(\tilde u_\delta[\rho]-[\rho u]\big)\Big(\big(\tilde u_\delta\phi(\tilde u_\delta)\big)'-\tilde u_\delta\phi'(\tilde u_\delta)-\phi(\tilde u_\delta)\Big)\\&=0.\end{aligned} \tag{51}$$

联立式(50)和(51)得:

$$\begin{aligned}&\lim_{\varepsilon\to\varepsilon_0^+}\int_{-\infty}^{+\infty}\big(\rho(\xi)-\rho_--[\rho]H(\xi-\tilde u_\delta)\big)\phi(\xi)\mathrm{d}\xi\\&=\big(\tilde u_\delta[\rho]-[\rho u]\big)\phi(\tilde u_\delta)=\rho_+(u_--u_+)\phi(\tilde u_\delta).\end{aligned} \tag{52}$$

同理,由式(48)得:

$$\begin{aligned}&\lim_{\varepsilon\to\varepsilon_0^+}\int_{-\infty}^{+\infty}\big(\rho(\xi)u(\xi)+\varepsilon u(\xi)-\rho_-u_--\varepsilon u_--[\rho u+\varepsilon u]H(\xi-\tilde u_\delta)\big)\phi(\xi)\mathrm{d}\xi\\&=\big(\tilde u_\delta[\rho u+\varepsilon u]-[\rho u^2+\varepsilon u^2]\big)\phi(\tilde u_\delta)\\&=\rho_+u_-(u_--u_+)\phi(\tilde u_\delta).\end{aligned} \tag{53}$$

考虑ρ和ρu的极限,对于$\psi\in C_0^\infty(R\times R^+)$,式(52)可转化为:

$$\begin{aligned}&\lim_{\varepsilon\to\varepsilon_0^+}\int_0^{+\infty}\int_{-\infty}^{+\infty}\left(\rho\left(\frac{x}{t}\right)-\rho_--[\rho]H(x-\tilde u_\delta t)\right)\psi(x,t)\mathrm{d}x\mathrm{d}t\\&=\lim_{\varepsilon\to\varepsilon_0^+}\int_0^{+\infty}\left(\int_{-\infty}^{+\infty}\big(\rho(\xi)-\rho_--[\rho]H(\xi-\tilde u_\delta)\big)\psi(\xi t,t)\mathrm{d}\xi\right)t\mathrm{d}t\\&=\int_0^{+\infty}\rho_+(u_--u_+)t\cdot\psi(u_-t,t)\mathrm{d}t.\end{aligned} \tag{54}$$

类似的,有:

$$\begin{aligned}&\lim_{\varepsilon\to\varepsilon_0^+}\int_0^{+\infty}\int_{-\infty}^{+\infty}\left(\rho\left(\frac{x}{t}\right)u\left(\frac{x}{t}\right)-\varepsilon u\left(\frac{x}{t}\right)-\rho_-u_--[\rho u]H(x-\tilde u_\delta t)\right)\psi(x,t)\mathrm{d}x\mathrm{d}t\\&=\lim_{\varepsilon\to\varepsilon_{0^+}}\int_0^{+\infty}\left(\int_{-\infty}^{+\infty}\big(\rho(\xi)u(\xi)-\varepsilon u(\xi)\big)-\rho_-u_--[\rho u]H(x-\tilde u_\delta)\psi(\xi t,t)\mathrm{d}\xi\right)t\mathrm{d}t\\&=\int_0^{+\infty}\rho_+u_-(u_--u_+)t\cdot\psi(u_-t,t)\mathrm{d}t.\end{aligned} \tag{55}$$

定理证毕.

定理 3　当$0 < u_+ < u_-$、$0 < \varepsilon < \varepsilon_0 = \dfrac{\rho_+(u_- - u_+)}{u_+}$时，黎曼问题(3)和(4)的狄拉克激波解收敛到输运方程(5)的狄拉克激波解.

证明　由式(12~13)和(33~34)得：

当$\rho_+ \neq \rho_-$时，

$$\begin{cases} \lim\limits_{\varepsilon \to 0^+} \tilde{u}_\delta = \dfrac{\sqrt{\rho_-}\, u_- + \sqrt{\rho_+}\, u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}} = u_\delta, \\ \lim\limits_{\varepsilon \to 0^+} \tilde{w}(t) = \sqrt{\rho_- \rho_+}\,(u_- - u_+)t = w(t). \end{cases} \tag{56}$$

当$\rho_+ = \rho_-$时，

$$\begin{cases} \lim\limits_{\varepsilon \to 0^+} \tilde{u}_\delta = \dfrac{u_+ + u_-}{2} = u_\delta, \\ \lim\limits_{\varepsilon \to 0^+} \tilde{w}(t) = \rho_+(u_- - u_+)t = w(t). \end{cases} \tag{57}$$

定理 4　若$u_+ > u_-$，当$\varepsilon \to 0^+$时，黎曼问题(3)和(4)的解$(J_1 + R_2)$收敛到黎曼问题(4)和(5)的真空解.

证明　由式(27)得：

$$\lim_{\varepsilon \to 0^+} \rho_* = \lim_{\varepsilon \to 0^+} \frac{\varepsilon \rho_+ u_-}{\varepsilon u_+ + \rho_+(u_+ - u_-)} = 0, \quad \lim_{\varepsilon \to 0^+} u_* = u_-. \tag{58}$$

由式(28)得：

$$\begin{cases} \lim\limits_{\varepsilon \to 0^+} \lambda_2(\rho_*, u_*) = \lim\limits_{\varepsilon \to 0^+} \left(1 + \dfrac{\varepsilon u_+ + \rho_+(u_+ - u_-)}{\varepsilon u_+ + \rho_+ u_+}\right) u_- = \left(2 - \dfrac{u_-}{u_+}\right) u_-, \\ \lim\limits_{\varepsilon \to 0^+} \lambda_2(\rho_+, u_+) = \lim\limits_{\varepsilon \to 0^+} \left(1 + \dfrac{\varepsilon}{\rho_+ + \varepsilon}\right) u_+ = u_+. \end{cases} \tag{59}$$

由式(58~59)可知，当$\varepsilon \to 0^+$时，稀疏波R_2的波后在直线$x = \left(2 - \dfrac{u_-}{u_+}\right) u_- t$处收敛到接触间断$\bar{J}$，波前在直线$x = u_- t$处收敛到接触间断$J_2$，接触间断$J_1$和稀疏波$R_2$之间的中间状态转变为真空态(图3).

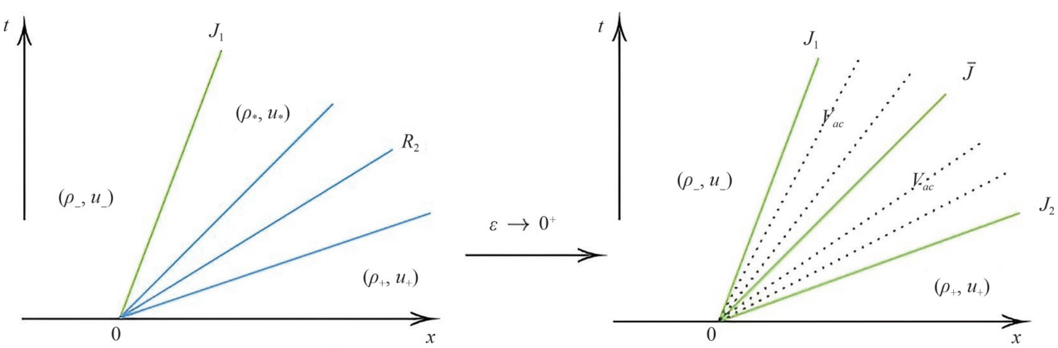

图 3　物理平面(x, t)

Figure 3　The physical plane(x, t)

当$\varepsilon \to 0^+$时，稀疏波R_2的内部状态转换为相应的真空状态. 任取$\xi \in \big(\lambda_2(\rho_*, u_*), \lambda_2(\rho_+, u_+)\big)$，由式(27)得：

$$\left(1 + \frac{\varepsilon u_+ + \rho_+(u_+ - u_-)}{\varepsilon u_+ + \rho_+ u_+}\right) u_- = \lambda_2(\rho_*, u_*) < \xi < \lambda_2(\rho_+, u_+) = \left(1 + \frac{\varepsilon}{\rho_+ + \varepsilon}\right) u_+, \tag{60}$$

由式(28)得：

$$\xi=\left(2-\frac{u\rho_+}{(\varepsilon+\rho_+)u_+}\right)u, \tag{61}$$

令 $\lim\limits_{\varepsilon\to 0^+}u=\bar{u}$. 当 $\varepsilon\to 0^+$ 时,由式(61)得 $\lim\limits_{\varepsilon\to 0^+}\xi=\left(2-\frac{\bar{u}}{u_+}\right)\bar{u}$.

当 $\varepsilon\to 0^+$ 时,特征线 $x=\xi t$ 位于稀疏波 R_2 的内部,式(60)可化简为:

$$\left(2-\frac{u_-}{u_+}\right)u_-<\xi=\left(2-\frac{\bar{u}}{u_+}\right)\bar{u}<\left(2-\frac{u_+}{u_+}\right)u_+, \tag{62}$$

引入函数:

$$f(\bar{u})=\left(2-\frac{\bar{u}}{u_+}\right)\bar{u},\quad 0<u_-<\bar{u}<u_+, \tag{63}$$

当 $\bar{u}\in(u_-,\ u_+)$ 时,得到 $f'(\bar{u})=2\left(1-\frac{\bar{u}}{u_+}\right)>0$,说明函数 $f(\bar{u})$ 在区间 (u_-,u_+) 内严格单调递增. 故:

$$u_-<\bar{u}<u_+, \tag{64}$$

由式(28)得:

$$\rho=\frac{\varepsilon\rho_+u}{\varepsilon u_++\rho_+(u_+-u)}, \tag{65}$$

由式(64)得:

$$0<\lim_{\varepsilon\to 0^+}\rho_+(u_+-u_-)+\varepsilon u_+=\rho_+(u_+-\bar{u})<\rho_+(u_+-u_-), \tag{66}$$

故:

$$\lim_{\varepsilon\to 0^+}\rho=\frac{\varepsilon\rho_+u}{\varepsilon u_++\rho_+(u_+-u)}=0, \tag{67}$$

当 $\varepsilon\to 0^+$ 时,稀疏波 R_2 内部的状态收敛到真空状态.

4 数值模拟

为证实第3节黎曼问题(3)和(4)解中 δ 激波和真空的形成,本节给出具体数值模拟,以验证前文建立的理论分析. 采用分裂系数矩阵方法(Splitting Coefficient Matrix Method)[30]的一阶迎风格式,式(3)可重写为:

$$\boldsymbol{U}_t+\boldsymbol{B}\boldsymbol{U}_x=0, \tag{68}$$

其中:

$$\boldsymbol{U}=(\rho,u)^T,\quad \boldsymbol{B}=\begin{pmatrix}u & \rho\\ 0 & u+\dfrac{\varepsilon u}{\rho+\varepsilon}\end{pmatrix}, \tag{69}$$

矩阵 $\boldsymbol{B}$ 可被改写为 $\boldsymbol{B}=\boldsymbol{R}\boldsymbol{\Lambda}\boldsymbol{L}$ 形式,其中:

$$\boldsymbol{R}=\begin{pmatrix}1 & \dfrac{\rho+\varepsilon}{\varepsilon u}\\ 0 & \dfrac{1}{\rho}\end{pmatrix},\quad \boldsymbol{\Lambda}=\begin{pmatrix}u & 0\\ 0 & u+\dfrac{\varepsilon u}{\rho+\varepsilon}\end{pmatrix},\quad \boldsymbol{L}=\begin{pmatrix}1 & -\dfrac{\rho(\rho+\varepsilon)}{\varepsilon u}\\ 0 & \rho\end{pmatrix}, \tag{70}$$

基于SCMM的一阶迎风格式可表示为:

$$\boldsymbol{U}_j^{n+1}=\boldsymbol{U}_j^n-\frac{\Delta t}{\Delta x}\left\{\boldsymbol{B}_j^-\left(\boldsymbol{U}_{j+1}^n-\boldsymbol{U}_j^n\right)+\boldsymbol{B}_j^+\left(\boldsymbol{U}_j^n-\boldsymbol{U}_{j-1}^n\right)\right\}, \tag{71}$$

其中:

$$\boldsymbol{B}_j^-=\frac{\boldsymbol{B}_j^n-\left|\boldsymbol{B}_j^n\right|}{2},\quad \boldsymbol{B}_j^+=\frac{\boldsymbol{B}_j^n+\left|\boldsymbol{B}_j^n\right|}{2},\quad \left|\boldsymbol{B}_j^n\right|=\boldsymbol{R}_j^n\left|\boldsymbol{\Lambda}_j^n\right|\boldsymbol{L}_j^n. \tag{72}$$

情形1　$0 < u_+ < u_-$.

对于该情形，由第3节可知，存在$\varepsilon_0 = \dfrac{\rho_+(u_- - u_+)}{u_+}$，使得$\varepsilon > \varepsilon_0$时，黎曼问题(3)和(4)的解为$(J_1 + S_2)$．当$\varepsilon_0 > \varepsilon > 0$时，黎曼问题(3)和(4)的解为$\delta S$．为更好观察$\varepsilon$趋于0时$\delta S$的形成，取如下初值：

$$(\rho, u)(x, t = 0) = \begin{cases} (0.6, 1.2), & -\infty < x < 0, \\ (0.4, 0.8), & 0 < x < +\infty, \end{cases} \tag{73}$$

那么，由初值(73)得$\varepsilon_0 = 0.2$.

由图4可知，当ε减小到$\varepsilon_0 = 0.2$时，中间状态密度ρ_*急剧增加，出现集中现象．当ε最终趋于0时，ρ_*的集中导致在极限处产生狄拉克激波．总之，随着压力减小，黎曼问题(3)和(4)的解$(J_1 + S_2)$收敛到黎曼问题(3)和(4)的δS．且当ε最终趋于0时，该δS收敛到黎曼问题(4)和(5)的δS．这与第3节分析的情形和产生的理论一致.

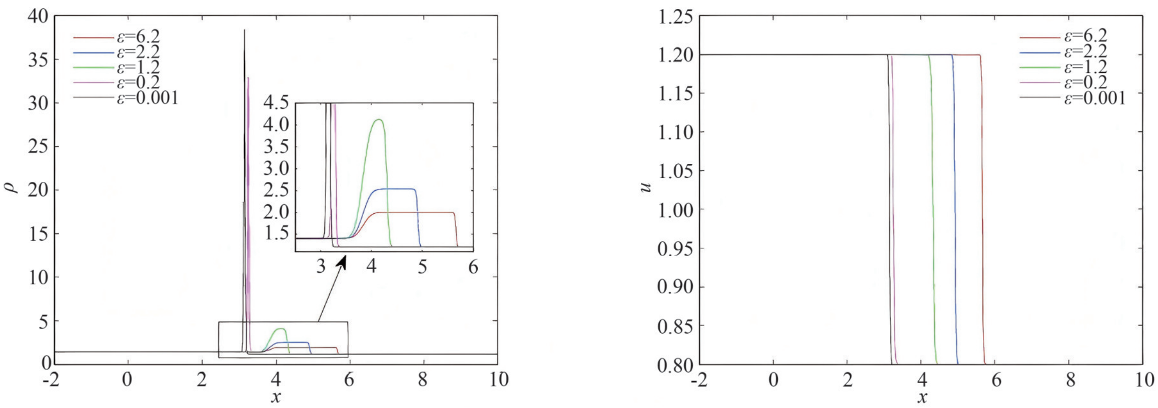

图4　当$\varepsilon = 0.001$、$\varepsilon = 0.2$、$\varepsilon = 1.2$、$\varepsilon = 2.2$、$\varepsilon = 6.2$且$t = 3$时带有初始条件(73)的ρ和u的数值结果

Figure 4　The numerical results of ρ and u with the initial condition of (73) at $t = 3$ when $\varepsilon = 0.001$, $\varepsilon = 0.2$, $\varepsilon = 1.2$, $\varepsilon = 2.2$, $\varepsilon = 6.2$

情形2　$0 < u_- < u_+$．初始条件选择如下：

$$(\rho, u)(x, t = 0) = \begin{cases} (0.25, 0.8), & -\infty < x < 0, \\ (0.15, 2.4), & 0 < x < +\infty. \end{cases} \tag{74}$$

由图5可知，随着ε逐渐减少且趋于0，中间状态密度ρ_*无限趋于0，即黎曼问题(3)和(4)的解$(J_1 + R_2)$收敛到黎曼问题(4)和(5)的真空解，与第3节分析的情形和产生的理论一致.

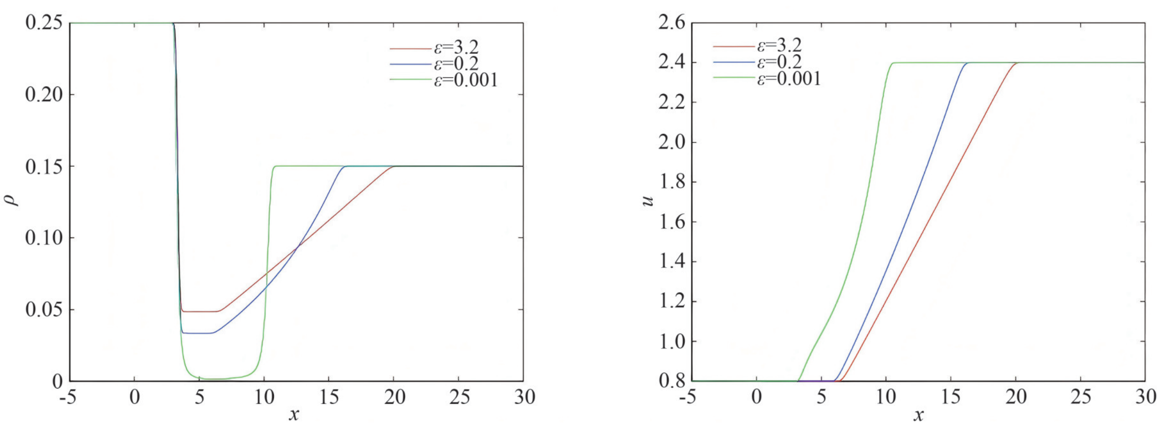

图5　当$\varepsilon = 0.001$、$\varepsilon = 0.2$、$\varepsilon = 3.2$且$t = 5$时带有初始条件(74)的ρ和u的数值结果

Figure 5　The numerical results of ρ and u with the initial condition of (74) at $t = 5$ when $\varepsilon = 0.001$, $\varepsilon = 0.2$, $\varepsilon = 3.2$

5 结 论

本文主要研究预期因子为反 Chaplygin 气体宏观生产模型黎曼解的极限行为．第一部分研究了反 Chaplygin 气体宏观生产模型的黎曼问题,得到3种黎曼解．当$0 < u_- < u_+$时,黎曼问题(3)和(4)的解为$(J_1 + R_2)$；当$0 < \frac{\rho_+ u_-}{\rho_+ + \varepsilon} < u_+ < u_-$时,黎曼问题(3)和(4)的解为$(J_1 + S_2)$；当$0 < u_+ \leqslant \frac{\rho_+ u_-}{\rho_+ + \varepsilon} < u_-$时,黎曼问题(3)和(4)的解为$\delta S$．第二部分研究了反 Chaplygin 气体宏观生产模型的压力消失极限．当扰动参数ε减小到仅依赖初值的参数ε_0时,证明了黎曼问题(3)和(4)的解$(J_1 + S_2)$收敛到黎曼问题(3)和(4)的δS．且当ε最终趋于0时,该δS收敛到黎曼问题(4)和(5)的δS;此外,还证明了在$\varepsilon \to 0$时黎曼问题(3)和(4)的解$(J_1 + R_2)$收敛到黎曼问题(4)和(5)的真空解．最后,给出了具有代表性的数值实验结果,与理论结果一致．

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